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Noncomputability in Models of Physical Phenomena

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We consider first the three-dimensional wave equation. It is well known that the solution $u(x, y, z, t)$ is uniquely determined by the initial conditions u and $\partial u/\partial t$ at time $t = 0$. We ask whether computable initial data can give rise to noncomputable solutions. The answer is "yes." Theorem 2 below gives an example in which the solution $u(x, y, z, t)$ takes a noncomputable value at a computable point in space-time. Theorem 3 is more delicate. It provides an example in which the solution maps each computable sequence of points into a computable sequence. Thus an experimenter, carrying out an arbitrary sequence of experiments programmed in advance, would obtain a computable sequence as his values. Nevertheless the function u is not computable. We note in passing that "computability," as used in this paper, is a technical term familiar to logicians from recursion theory. The precise definitions are spelled out below.

The noncomputable solutions discussed above are of the type commonly called "weak solutions," i.e., although continuous, they are not twice differentiable at all points. Weak solutions describe shock waves and other nondifferentiable patterns which frequently appear in physical phenomena. The use of weak solutions is inevitable, for it can be proved that no C^2 solutions of the desired kind exist (Pour-El and Richards, 1981).

We review here the standard definitions of computability as they apply to classical analysis. The notion of a computable or "recursive" function from the nonnegative integers into themselves will be assumed as known. Then a sequence $\{r_n\}$ of rational numbers is called "recursive" if there exist three recursive functions $a(n)$, $b(n)$, $s(n)$ such that $r_n = (-1)^{s(n)}[a(n)/b(n)]$.

Definition. A real number x is called *computable* if there exists a recursive sequence of rationals $\{r_n\}$ which converges *effectively* to x; this means that there is a recursive function $e(n)$ such that

$$k \geq e(n) \qquad \text{implies } |x - r_k| \leq 10^{-n}$$

By passing effectively to the subsequence $\{r'_n\} = \{r_{e(n)}\}$, we have

$$|x - r'_n| \leq 10^{-n} \qquad \text{for all } n$$

Definition. A sequence of real numbers $\{x_k\}$ is called *computable* if there is a recursive double sequence of rationals $\{r_{kn}\}$ such that

$$|x_k - r_{kn}| \leq 10^{-n} \qquad \text{for all } k, n$$

Now we come to the notion of a computable function of q real variables. For our purposes, we can consider a real-valued function f defined on a closed bounded rectangle $I^q = \{a_1 \leq x_1 \leq b_1, \ldots, a_q \leq x_q \leq b_q\}$ with computable end points. This definition has been given in several equivalent forms (Grzegorczyk, 1955, 1957; Pour-El and Caldwell, 1975). One of these, which is useful in applications, is the following:

A function f from a computable closed bounded rectangle I^q in $\mathbb{R}^q$ into $\mathbb{R}^1$ is called *computable* if (a) f is *sequentially computable*, i.e., for every computable sequence $\{\mathbf{x}_k\}$ of points in I^q, the sequence of values $\{f(\mathbf{x}_k)\}$ is computable; and (b) f is *effectively uniformly continuous*, i.e., there exists a recursive function $d(n)$ such that, for all points $\mathbf{x}, \mathbf{y} \in I^q$,

$$|\mathbf{x} - \mathbf{y}| \leq 1/d(n) \qquad \text{implies } |f(\mathbf{x}) - f(\mathbf{y})| \leq 10^{-n}$$

We now state our main results (Pour-El and Richards, 1979, 1981). The first is concerned with ordinary differential equations.

> *Theorem 1.* There exists a computable, and hence continuous, function f such that the equation
>
> $$\frac{dy}{dx} = f(x, y)$$
>
> has no computable solution in any rectangle, no matter how small, within the domain of definition of f.

As stated above, the next two theorems are concerned with the three-dimensional wave equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = 0 \qquad (1a)$$

with initial conditions

$$u(x, y, z, 0) = f(x, y, z)$$
$$\frac{\partial u}{\partial t}(x, y, z, 0) = 0 \tag{1b}$$

It is well known that if f is of class C^1, then (1a, b) has a unique solution. The following results show that computable initial data can give rise to noncomputable solutions in two ways.

Theorem 2. There exists a computable, and hence continuous, function f such that the solution u of (1), although continuous, is not computable. In fact $u(0,0,0,1)$ is a noncomputable real.

Theorem 3. There exists a computable function f such that the solution u of (1) is continuous, and (i) u is sequentially computable [cf. (a)] and (ii) $u(x, y, z, 1)$ is not a computable function of x, y, z.

The last two theorems extend to all space dimensions $n \geq 2$.

The following brief comments on the methods of proof may be in order. Obviously the constructions must involve a synthesis of recursion theory and classical analysis. Furthermore, in order to prove functions of a real variable noncomputable, it is necessary to maintain a tight control over the analytic processes employed, so that inputs borrowed from recursion theory can be traced through each step of the construction.

Theorems 2 and 3 provide examples in which computable data are transformed by a physical process into data which are no longer computable.

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